

Leading and next to leading large n_f terms in the cusp anomalous dimension and the quark–antiquark potential

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I discuss 3 related quantities: the cusp anomalous dimension, the HQET heavy-quark field anomalous dimension, and the quark–antiquark potential. Leading large n_f terms can be calculated to all orders in α_s . Next to leading terms with the abelian color structure C_F^2 also can be found to all orders (but not non-abelian $C_F C_A$ terms). This talk is based on Appendices C and D in [1].

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Figure 1: The light-by-light diagram is $n_f \alpha^4$, and hence $\text{NNL}\beta_0$.

With the $\text{NL}\beta_0$ accuracy the renormalization constant Z of the heavy-to-heavy current (the cusp) is given by

$$\log W(t, t'; \varphi) - \log W(t, t'; 0) = \text{[cusp diagram]} - \text{[self-energy diagram]} = \log Z + \text{finite} \quad (2.3)$$

(diagrams where both photon-interaction vertices are before the cusp, or after the cusp, cancel in this difference). Going to momentum space, we can express it via the vertex function $V(\omega, \omega'; \varphi)$ (it is convenient to set $\omega' = \omega$, in order to have a single-scale problem):

$$V(\omega, \omega; \varphi) - V(\omega, \omega; 0) = \text{[cusp diagram]} - \text{[self-energy diagram]} = \log Z + \text{finite}. \quad (2.4)$$

The HQET field renormalization can be obtained from $V(\omega, \omega; 0)$.

The static quark–antiquark potential can be considered similarly. The terms with the highest power of n_f in each order of perturbation theory have the structures $C_F(T_F n_f)^L \alpha_s^{L+1}$ ($L \geq 0$). The terms with next to highest power of n_f have the structures $C_F^2(T_F n_f)^{L-1} \alpha_s^{L+1}$ and $C_F C_A(T_F n_f)^{L-1} \alpha_s^{L+1}$ ($L \geq 2$); we'll consider only the abelian ones. In the Coulomb gauge, up to $\text{NL}\beta_0$ the potential is given by the full Coulomb photon propagator

$$V(\vec{q}) = \text{[Coulomb diagram]} = -\frac{e_0^2}{\vec{q}^2} \frac{1}{1 - \Pi(-\vec{q}^2)} \quad (2.5)$$

($\Pi(q^2)$ is gauge invariant in QED, and can be taken from covariant-gauge calculations). This simple equality is first broken at $\text{NNL}\beta_0$ order by the light-by-light diagram (figure 2).

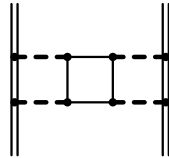


Figure 2: The light-by-light diagram is $n_f \alpha^4$, and hence $\text{NNL}\beta_0$.

As discussed in [1], conformal symmetry leads to the relation between $\Gamma(\pi - \delta)$ at $\delta \rightarrow 0$ and $V(\vec{q})$:

$$\Delta \equiv [\delta \Gamma(\pi - \delta; \alpha_s)]_{\delta \rightarrow 0} - \frac{\vec{q}^2 V(\vec{q}; \alpha_s)}{4\pi} = 0 \quad (2.6)$$

(this relation has been observed in [21] at 2 loops). In QCD (and QED) conformal symmetry is anomalous (thus leading to non-zero β function), and [1]

$$\Delta = \frac{\pi}{108} \beta_0 C_F \left(\frac{\alpha_s}{\pi} \right)^3 (47C_A - 28T_F n_f) + \mathcal{O}(\alpha_s^4). \quad (2.7)$$

3. Leading β_0 order

The photon self energy at the $L\beta_0$ order is ~ 1 :

$$\begin{aligned} \Pi_0(k^2) &= \text{diagram} = \beta_0 \frac{e_0^2}{(4\pi)^{d/2}} e^{-\gamma\epsilon} \frac{D(\epsilon)}{\epsilon} (-k^2)^{-\epsilon}, \\ D(\epsilon) &= e^{\gamma\epsilon} \frac{(1-\epsilon)\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(1-2\epsilon)(1-\frac{2}{3}\epsilon)\Gamma(1-2\epsilon)} = 1 + \frac{5}{3}\epsilon + \dots \end{aligned} \quad (3.1)$$

The charge renormalization in the $\overline{\text{MS}}$ scheme is

$$\beta_0 \frac{e_0^2}{(4\pi)^{d/2}} e^{-\gamma\epsilon} = b Z_\alpha(b) \mu^{2\epsilon}. \quad (3.2)$$

At the $L\beta_0$ order we can solve the RG equation

$$\frac{d \log Z_\alpha}{d \log b} = -\frac{b}{\epsilon + b}$$

and obtain

$$Z_\alpha = \frac{1}{1 + b/\epsilon}. \quad (3.3)$$

The vertex $V(\omega, \omega; \varphi)$ is given by the one-loop diagram with the factor $1/(1 - \Pi(k^2))$ inserted in the integrand. At the $L\beta_0$ order (figure 3) the result can be written in the form

$$V(\omega, \omega; \varphi) = \text{diagram} = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{f(\epsilon, L\epsilon; \varphi)}{L} \Pi_0^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \quad (3.4)$$

where L is the number of loops and Π_0 (3.1) is taken at $-k^2 = (-2\omega)^2$. Reduction of such integrals to master ones, as well as evaluation of these master integrals, has been considered in [22]. In

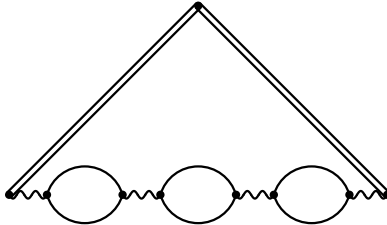


Figure 3: The L -loop vertex diagram at the $L\beta_0$ order contains $L - 1$ Π_0 insertions.

Landau gauge we obtain

$$f(\varepsilon, u; \varphi) = -\frac{(1 - \frac{2}{3}\varepsilon)\Gamma(2 - 2\varepsilon)\Gamma(1 - u)\Gamma(1 + 2u)}{(1 - \varepsilon)\Gamma^2(1 - \varepsilon)\Gamma(1 + \varepsilon)\Gamma(2 + u - \varepsilon)} \times \left[((2 + u - 2\varepsilon)\cos\varphi - u) {}_2F_1\left(\begin{matrix} 1, 1 - u \\ 3/2 \end{matrix} \middle| \frac{1 - \cos\varphi}{2}\right) + 1 \right] \quad (3.5)$$

(in an arbitrary covariant gauge, a one-loop gauge-dependent contribution should be added). The function $f(\varepsilon, u; \varphi)$ is regular at the origin:

$$f(\varepsilon, u; \varphi) = \sum_{n,m=0}^{\infty} f_{nm}(\varphi) \varepsilon^n u^m. \quad (3.6)$$

The renormalization constant Z can be written as

$$\log Z = \frac{Z_1}{\varepsilon} + \frac{Z_2}{\varepsilon^2} + \dots, \quad Z_n = \mathcal{O}(b^n).$$

Only Z_1 is needed in order to obtain

$$\Gamma(b; \varphi) = -2 \frac{dZ_1(b; \varphi)}{d \log b};$$

higher Z_n contain no new information, and are uniquely reconstructed from Z_1 using self-consistency conditions. Choosing

$$\mu^2 = D(\varepsilon)^{-1/\varepsilon} (-2\omega)^2 \rightarrow e^{-\frac{5}{3}\varepsilon} (-2\omega)^2$$

we have

$$V(\omega, \omega; \varphi) - V(\omega, \omega; 0) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{\bar{f}(\varepsilon, L\varepsilon; \varphi)}{L} \left(\frac{b}{\varepsilon + b} \right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \quad (3.7)$$

where $\bar{f}(\varepsilon, u; \varphi) = f(\varepsilon, u; \varphi) - f(\varepsilon, u; 0)$. We expand in b , expand $\bar{f}(\varepsilon, u; \varphi)$ in ε and u and select only ε^{-1} terms in order to obtain Z_1 . All coefficients but f_{n0} cancel:

$$Z_1(b; \varphi) = 2 \frac{\varphi \cot \varphi - 1}{\beta_0} \sum_{n=0}^{\infty} \frac{\hat{f}_n}{n+1} (-b)^{n+1},$$

where

$$\bar{f}(\varepsilon, 0; \varphi) = -2\hat{f}(\varepsilon)(\varphi \cot \varphi - 1), \quad \hat{f}(\varepsilon) = \sum_{n=0}^{\infty} \hat{f}_n \varepsilon^n.$$

Therefore at the $L\beta_0$ we obtain [23]

$$\begin{aligned} \Gamma(b; \varphi) &= 4 \frac{b}{\beta_0} \gamma_0(b)(\varphi \cot \varphi - 1) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \\ \gamma_0(b) &= \hat{f}(-b) = \frac{(1 + \frac{2}{3}b)\Gamma(2 + 2b)}{(1 + b)\Gamma^3(1 + b)\Gamma(1 - b)} \\ &= 1 + \frac{5}{3}b - \frac{1}{3}b^2 - \left(2\zeta_3 - \frac{1}{3}\right)b^3 + \left(\frac{\pi^4}{30} - \frac{10}{3}\zeta_3 - \frac{1}{3}\right)b^4 + \dots \end{aligned} \quad (3.8)$$

As a free bonus, we can obtain the HQET field anomalous dimension. The vertex function V at $\varphi = 0$ is related to the HQET propagator S by the Ward identity

$$V(\omega, \omega'; 0) = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega}, \quad V(\omega, \omega; 0) = \frac{dS^{-1}(\omega)}{d\omega}. \quad (3.9)$$

Therefore the renormalization constant of the HQET quark field Z_h is given by

$$\log V(\omega, \omega'; 0) = -\log Z_h + \text{finite}.$$

Using

$$f(\varepsilon, u; 0) = -3 \frac{(1 - \frac{2}{3}\varepsilon)^2 \Gamma(2 - 2\varepsilon) \Gamma(1 - u) \Gamma(1 + 2u)}{(1 - \varepsilon) \Gamma^2(1 - \varepsilon) \Gamma(1 + \varepsilon) \Gamma(2 + u - \varepsilon)},$$

we obtain in the Landau gauge [24]

$$\begin{aligned} \gamma_h(b) &= 2 \frac{b}{\beta_0} \gamma_{h0}(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \\ \gamma_{h0}(b) &= f(-b, 0; 0) = \frac{(1 + \frac{2}{3}b)^2 \Gamma(2 + 2b)}{(1 + b)^2 \Gamma^3(1 + b) \Gamma(1 - b)} \\ &= 1 + \frac{4}{3}b - \frac{5}{9}b^2 - \left(2\zeta_3 - \frac{2}{3}\right)b^3 + \left(\frac{\pi^4}{30} - \frac{8}{3}\zeta_3 - \frac{7}{9}\right)b^4 + \dots \end{aligned} \quad (3.10)$$

(in an arbitrary covariant gauge, a one-loop gauge-dependent contribution should be added).

Now we consider the potential $V(\vec{q})$ at the $L\beta_0$ order. Choosing $\mu^2 = \vec{q}^2$ we have

$$V(\vec{q}) = -\frac{(4\pi)^{D/2} e^{\gamma_E}}{\beta_0 D(\varepsilon) (\vec{q}^2)^{1-\varepsilon}} \varepsilon \sum_{L=1}^{\infty} \left(D(\varepsilon) \frac{b}{\varepsilon + b} \right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right).$$

The sum here can be written as

$$\sum_{L=1}^{\infty} g(\varepsilon, L\varepsilon) \left(\frac{b}{\varepsilon + b} \right)^L, \quad g(\varepsilon, u) = D(\varepsilon)^{u/\varepsilon} = \sum_{n,m=0}^{\infty} g_{nm} \varepsilon^n u^m.$$

This sum is equal to

$$\frac{b}{\varepsilon} \sum_{n=0}^{\infty} n! g_{0n} b^n + \mathcal{O}(\varepsilon^0)$$

($1/\varepsilon^n$ terms with $n > 1$ vanish, so that $V(\vec{q})$ is automatically finite), where

$$g(0, u) = e^{\frac{5}{3}u}, \quad g_{0n} = \frac{1}{n!} \left(\frac{5}{3} \right)^n. \quad (3.11)$$

Therefore

$$V(\vec{q}) = -\frac{(4\pi)^2}{\vec{q}^2} \frac{b}{\beta_0} V_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \quad V_0(b) = \frac{1}{1 - \frac{5}{3}b}. \quad (3.12)$$

The conformal anomaly (2.6) at the $L\beta_0$ order is

$$\begin{aligned} \Delta &= 4\pi \frac{b^3}{\beta_0} \delta_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \\ \delta_0(b) &= \frac{V_0(b) - \gamma_0(b)}{b^2} = \frac{28}{9} + 2 \left(\zeta_3 + \frac{58}{27} \right) b - \frac{1}{3} \left(\frac{\pi^4}{10} - 10\zeta_3 - \frac{652}{27} \right) b^2 + \dots \end{aligned} \quad (3.13)$$

The first term here reproduces the T_{Fn_f} term in (2.7).

4. Next to leading β_0 order

To obtain the photon propagator with the $\text{NL}\beta_0$ accuracy, we need the photon self-energy up to $1/\beta_0$:

$$\Pi(k^2) = \text{bubble} + 2 \text{triangle} + \text{triangle} = \Pi_0(k^2) + \frac{\Pi_1(k^2)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \quad (4.1)$$

where the photon propagators in Π_1 are taken at the $L\beta_0$ order. The $\text{NL}\beta_0$ contribution can be written in the form [25, 26]

$$\Pi_1(k^2) = 3\varepsilon \sum_{L=2}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \Pi_0(k^2)^L. \quad (4.2)$$

Using integration by parts, one can reduce it to

$$\begin{aligned} F(\varepsilon, u) &= \frac{2(1-2\varepsilon)^2(3-2\varepsilon)\Gamma^2(1-2\varepsilon)}{9(1-\varepsilon)(1-u)(2-u)\Gamma^2(1-\varepsilon)\Gamma^2(1+\varepsilon)} \\ &\times \left[-u \frac{2-3\varepsilon-\varepsilon^2+\varepsilon(2+\varepsilon)u-\varepsilon u^2}{\Gamma^2(1-\varepsilon)} I(1+u-2\varepsilon) \right. \\ &\quad \left. + 2 \frac{2(1+\varepsilon)(3-2\varepsilon)-(4+11\varepsilon-7\varepsilon^2)u+\varepsilon(8-3\varepsilon)u^2-\varepsilon u^3}{(1-u)(2-u)(1-u-\varepsilon)(2-u-\varepsilon)} \frac{\Gamma(1+u)\Gamma(1-u+\varepsilon)}{\Gamma(1-u-\varepsilon)\Gamma(1+u-2\varepsilon)} \right] \\ &= \sum_{n,m=0}^{\infty} F_{nm} \varepsilon^n u^m, \end{aligned} \quad (4.3)$$

where the integral

$$I(n) = \text{bubble}(n) = \frac{1}{\pi^d} \int \frac{d^d k_1 d^d k_2}{k_1^2 k_2^2 (k_1+p)^2 (k_2+p)^2 [(k_1-k_2)^2]^n}$$

(euclidean, $p^2 = 1$) can be expressed via a ${}_3F_2$ function of unit argument [27, 28] (see the review [29] for more references). The ${}_3F_2$ function can be expanded up to any desired order using known algorithms, the coefficients are expressed via multiple ζ values; therefore, the coefficients F_{nm} can be calculated to any desired order.

The function $F(\varepsilon, u)$ simplifies in some cases. In particular [25],

$$F(\varepsilon, 0) = \frac{(1+\varepsilon)(1-2\varepsilon)^2(1-\frac{2}{3}\varepsilon)^2\Gamma(1-2\varepsilon)}{(1-\varepsilon)^2(1-\frac{1}{2}\varepsilon)\Gamma(1+\varepsilon)\Gamma^3(1-\varepsilon)}, \quad (4.4)$$

so that F_{n0} contain no multiple ζ values, only ζ_n . Also [26]

$$F(0, u) = \frac{2}{3} \frac{\psi'(2-\frac{u}{2}) - \psi'(1+\frac{u}{2}) - \psi'(\frac{3-u}{2}) + \psi'(\frac{1+u}{2})}{(1-u)(2-u)} \quad (4.5)$$

so that F_{0m} contains only ζ_{2n+1} [26]:

$$F_{0m} = -\frac{32}{3} \sum_{s=1}^{[(m+1)/2]} s (1-2^{-2s}) (1-2^{2s-m-2}) \zeta_{2s+1} + \frac{4}{3} (m+1) (m+(m+6)2^{-m-3}). \quad (4.6)$$

The two-loop case is, of course, trivial:

$$F(\varepsilon, 2\varepsilon) = \frac{2}{9\varepsilon^2} \frac{3-2\varepsilon}{1-\varepsilon} \left[2 \frac{(1-2\varepsilon)^2(2-2\varepsilon+\varepsilon^2)}{(1-3\varepsilon)(2-3\varepsilon)} \frac{\Gamma(1+2\varepsilon)\Gamma^2(1-2\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(1-3\varepsilon)} - 2 + \varepsilon - 2\varepsilon^2 \right].$$

Let's write the charge renormalization constant Z_α with the $\text{NL}\beta_0$ accuracy as

$$\begin{aligned} Z_\alpha(b) &= \frac{1}{1+b/\varepsilon} \left[1 + \frac{Z_{\alpha 1}(b)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \right], \\ Z_{\alpha 1}(b) &= \frac{Z_{\alpha 11}(b)}{\varepsilon} + \frac{Z_{\alpha 12}(b)}{\varepsilon^2} + \dots, \quad Z_{\alpha 1n} = \mathcal{O}(b^{n+1}). \end{aligned} \quad (4.7)$$

In the abelian theory, $\log(1-\Pi)$ expressed (3.2) via renormalized b should be equal to $\log Z_\alpha + \text{finite}$. Equating the coefficients of ε^{-1} in the $1/\beta_0$ terms in this relation, we see that $Z_{\alpha 11}$ (4.7) is given by the coefficient of ε^{-1} in

$$-\left(1 + \frac{b}{\varepsilon}\right) \Pi_1.$$

It is convenient to choose

$$\mu^2 = D(\varepsilon)^{-1/\varepsilon} (-k^2) \rightarrow e^{-\frac{5}{3}\varepsilon} (-k^2),$$

then

$$\Pi_1 = 3\varepsilon \sum_{L=2}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \left(\frac{b}{\varepsilon+b} \right)^L.$$

We expand in b and expand $F(\varepsilon, u)$ in ε and u ; selecting ε^{-1} terms, we find that all coefficients but F_{n0} cancel:

$$Z_{\alpha 11} = -3 \sum_{n=0}^{\infty} \frac{F_{n0}(-b)^{n+2}}{(n+1)(n+2)}. \quad (4.8)$$

The β function with $\text{NL}\beta_0$ accuracy is

$$\beta(b) = b + \frac{\beta_1(b)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \quad (4.9)$$

where [25, 26]

$$\begin{aligned} \beta_1(b) &= -\frac{dZ_{\alpha 11}(b)}{d \log b} = 3 \sum_{n=0}^{\infty} \frac{F_{n0}(-b)^{n+2}}{n+1} \\ &= 3b^2 + \frac{11}{4}b^3 - \frac{77}{36}b^4 - \frac{1}{2} \left(3\zeta_3 + \frac{107}{48} \right) b^5 + \frac{1}{5} \left(\frac{\pi^4}{10} - 11\zeta_3 + \frac{251}{48} \right) b^6 + \dots \end{aligned} \quad (4.10)$$

(the coefficients F_{n0} follow from $F(\varepsilon, 0)$ (4.4)). The corresponding terms in the 5-loop QED β function [30] are reproduced. We shall need the full $Z_{\alpha 1}$, not just $Z_{\alpha 11}$; integrating the RG equation with the $1/\beta_0$ accuracy we obtain

$$Z_{\alpha 1}(b) = -\varepsilon \int_0^b \frac{\beta_1(b) db}{b(\varepsilon+b)^2} = -\frac{3}{2} \frac{b^2}{\varepsilon} + \frac{1}{2} (4 + F_{10}\varepsilon) \frac{b^3}{\varepsilon^2} - \frac{1}{4} (9 + 3F_{10}\varepsilon + F_{20}\varepsilon^2) \frac{b^4}{\varepsilon^3} + \dots$$

At the $\text{NL}\beta_0$ order we should expand the photon propagator $(1 - \Pi_0 - \Pi_1/\beta_0)^{-1}$ up to $1/\beta_0$ (Fig. 4). The vertex function (3.7) becomes

$$V(\omega, \omega; \varphi) - V(\omega, \omega; 0) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{\bar{f}(\varepsilon, L\varepsilon; \varphi)}{L} \left(\frac{b}{\varepsilon + b} \right)^L \times \left[1 + L \frac{Z_{\alpha 1}}{\beta_0} + \frac{3\varepsilon}{\beta_0} \sum_{L'=2}^{L-1} \frac{L-L'}{L'} F(\varepsilon, L'\varepsilon) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right), \quad (4.11)$$

where L' is the number of loops in the Π_1 insertion, and the $1/\beta_0$ correction $Z_{\alpha 1}$ to the charge renormalization (4.7) is taken into account. We expand in b and substitute the expansions (4.3) and (3.6); in Z_1 , the coefficient of ε^{-1} , all \bar{f}_{nm} except \bar{f}_{n0} cancel. At the $\text{NL}\beta_0$ order the cusp anomalous dimension is determined by the same \hat{f}_n coefficients as at the $L\beta_0$ order:

$$\Gamma(b; \varphi) = 4 \left[\frac{b}{\beta_0} \gamma_0(b) - \frac{b^3}{\beta_0^2} \gamma_1(b) \right] (\varphi \cot \varphi - 1) + \mathcal{O}\left(\frac{1}{\beta_0^3}\right), \quad (4.12)$$

where

$$\begin{aligned} \gamma_1(b) = & -\frac{3}{2} [F_{10} + 2F_{01} - 2\hat{f}_1] + [2F_{20} + 3(F_{11} + F_{02}) + 3F_{01}\hat{f}_1 - 6\hat{f}_2] b \\ & - \left[\frac{3}{4} (3F_{30} + 4(F_{21} + F_{12} + F_{03})) + (F_{20} + 3(F_{11} + F_{02}))\hat{f}_1 - \frac{3}{2} (F_{10} - 2F_{01})\hat{f}_2 - 9\hat{f}_3 \right] b^2 + \dots \end{aligned}$$

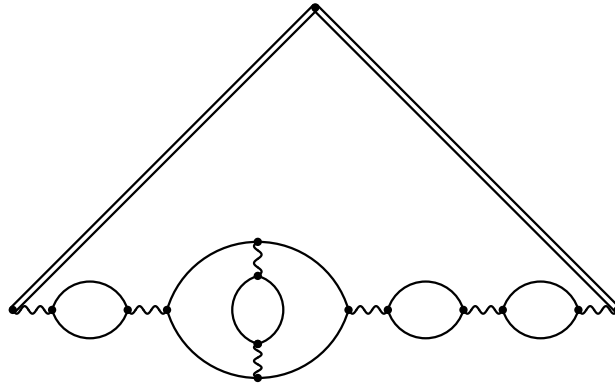


Figure 4: $\text{NL}\beta_0$ order diagrams contain one Π_1 insertion (with any number of Π_0 insertions inside) and any number of Π_0 insertions to the left and to the right of it.

Substituting F_{nm} we obtain

$$\begin{aligned}
\gamma_1(b) = & 12\zeta_3 - \frac{55}{4} + \left(-\frac{\pi^4}{5} + 40\zeta_3 - \frac{299}{18}\right)b \\
& + \left(24\zeta_5 - \frac{2}{3}\pi^4 + \frac{233}{6}\zeta_3 + \frac{15211}{864}\right)b^2 \\
& + \left(-48\zeta_3^2 - \frac{2}{63}\pi^6 + 80\zeta_5 - \frac{167}{225}\pi^4 + \frac{1168}{15}\zeta_3 - \frac{971}{240}\right)b^3 \\
& + \left(36\zeta_7 + \frac{8}{5}\pi^4\zeta_3 - 160\zeta_3^2 - \frac{20}{189}\pi^6 + \frac{377}{3}\zeta_5 - \frac{23}{15}\pi^4 + \frac{929}{12}\zeta_3 - \frac{8017}{1728}\right)b^4 \\
& + \left(-240\zeta_3\zeta_5 - \frac{4}{225}\pi^8 + 120\zeta_7 + \frac{16}{3}\pi^4\zeta_3 - \frac{2776}{21}\zeta_3^2 - \frac{914}{3969}\pi^6 \right. \\
& \left. + \frac{6826}{21}\zeta_5 - \frac{1793}{1350}\pi^4 - \frac{31693}{315}\zeta_3 + \frac{79433}{4320}\right)b^5 + \dots
\end{aligned} \tag{4.13}$$

This expansion can be extended to any number of loops. The first term in (4.13) agrees with the $C_F^2 T_F n_f$ term in the three-loop result [5, 6, 1]. The next term coincides with the $C_F^2 (T_F n_f)^2 \alpha_s^4$ term in Γ recently calculated in [31]. Note that the last (8-loop) term here contains F_{nm} with $n+m=6$, $n>0$, $m>0$, which contain $\zeta_{5,3}$; but they enter as the combination $F_{51} + F_{42} + F_{33} + F_{24} + F_{15}$ in which this $\zeta_{5,3}$ cancels.

Similarly, the field anomalous dimension in Landau gauge at the $NL\beta_0$ order is

$$\begin{aligned}
\gamma_h(b) = & -6 \left[\frac{b}{\beta_0} \gamma_{h0}(b) - \frac{b^3}{\beta_0^2} \gamma_{h1}(b) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right), \\
\gamma_{h1}(b) = & 3 \left(4\zeta_3 - \frac{17}{4} \right) + \left(-\frac{\pi^4}{5} + 36\zeta_3 - \frac{103}{9} \right) b \\
& + \left(24\zeta_5 - \frac{3}{5}\pi^4 + \frac{59}{2}\zeta_3 + \frac{14579}{864} \right) b^2 \\
& + \left(-48\zeta_3^2 - \frac{2}{63}\pi^6 + 72\zeta_5 - \frac{44}{75}\pi^4 + \frac{3229}{45}\zeta_3 - \frac{5191}{540} \right) b^3 \\
& + \left(36\zeta_7 + \frac{8}{5}\pi^4\zeta_3 - 144\zeta_3^2 - \frac{2}{21}\pi^6 + 107\zeta_5 - \frac{946}{675}\pi^4 + \frac{9601}{180}\zeta_3 + \frac{22859}{8640} \right) b^4 \\
& + \left(-240\zeta_3\zeta_5 - \frac{4}{225}\pi^8 + 108\zeta_7 + \frac{24}{5}\pi^4\zeta_3 - \frac{664}{7}\zeta_3^2 - \frac{272}{1323}\pi^6 \right. \\
& \left. + \frac{18574}{63}\zeta_5 - \frac{119}{135}\pi^4 - \frac{6263}{63}\zeta_3 + \frac{16103}{1296} \right) b^5 + \dots
\end{aligned} \tag{4.14}$$

The first term here coincides with the $C_F^2 T_F n_f$ term in the three-loop result obtained by a direct calculation [13, 14]. The last term contains the same combination of F_{nm} with $n+m=6$, so that $\zeta_{5,3}$ cancels.

The static potential at the $NL\beta_0$ level is

$$\begin{aligned}
V(\vec{q}) = & -\frac{(4\pi)^2}{\beta_0 \vec{q}^2} \varepsilon \sum_{L=1}^{\infty} g(\varepsilon, L\varepsilon) \left(\frac{b}{\varepsilon+b} \right)^L \left[1 + L \frac{Z_{\alpha 1}}{\beta_0} + \frac{3\varepsilon}{\beta_0} \sum_{L'=2}^{L-1} \frac{L-L'}{L'} F(\varepsilon, L'\varepsilon) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right) \\
= & -\frac{(4\pi)^2}{\vec{q}^2} \left[\frac{b}{\beta_0} V_0(b) - \frac{b^3}{\beta_0^2} V_1(b) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)
\end{aligned} \tag{4.15}$$

where

$$V_1(b) = -\frac{3}{2}[F_{10} + 2F_{01} + 2g_{01}] + \frac{1}{2}[F_{20} - 6F_{02} - 6(F_{10} + 3F_{01})g_{01} - 30g_{02}]b \\ - \frac{1}{4}[F_{30} + 24F_{03} - 4(F_{20} + 12F_{02})g_{01} + 36(F_{10} + 4F_{01})g_{02} + 312g_{03}]b^2 + \dots$$

contains only the same coefficients g_{0n} (3.11) as the $L\beta_0$ result, and only F_{n0} and F_{0m} are involved (see (4.4–4.6)). We obtain

$$V_1(b) = 12\zeta_3 - \frac{55}{4} + \left(78\zeta_3 - \frac{7001}{72}\right)b + \left(60\zeta_5 + \frac{723}{2}\zeta_3 - \frac{147851}{288}\right)b^2 \\ + \left(770\zeta_5 + \frac{\pi^4}{200} + \frac{276901}{180}\zeta_3 - \frac{70418923}{25920}\right)b^3 \\ + \left(1134\zeta_7 + \frac{32297}{5}\zeta_5 + \frac{41}{1800}\pi^4 + \frac{402479}{60}\zeta_3 - \frac{1249510621}{77760}\right)b^4 \\ + \left(21735\zeta_7 + \frac{\zeta_3^2}{7} + \frac{\pi^6}{1323} + \frac{5911849}{126}\zeta_5 + \frac{41}{720}\pi^4 + \frac{48558187}{1512}\zeta_3 - \frac{10255708489}{93312}\right)b^5 + \dots \quad (4.16)$$

Thus we have reproduced the $C_F(T_F n_f)^2 \alpha_s^3$ and $C_F^2 T_F n_f \alpha_s^3$ terms in the two-loop potential [17], as well as the $C_F(T_F n_f)^3 \alpha_s^4$ and $C_F^2(T_F n_f)^2 \alpha_s^4$ terms in the three-loop one [18]. This expansion can be extended to any order; it contains only ζ_n because only F_{n0} and F_{0m} are present. Note the pattern of the highest weights in (4.16): 3, 3, 5, 5, 7, 7, whereas one would expect 3, 4, 5, 6, 7, 8, as in (4.13), (4.14). The conformal anomaly (2.6) at the $NL\beta_0$ order is

$$\Delta = 4\pi \left[\frac{b^3}{\beta_0} \delta_0(b) - \frac{b^4}{\beta_0^2} \delta_1(b) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right), \\ \delta_1(b) = \frac{\pi^4}{5} + 38\zeta_3 - \frac{645}{8} + \left(36\zeta_5 + \frac{2}{3}\pi^4 + \frac{968}{3}\zeta_3 - \frac{114691}{216}\right)b \\ + \left(48\zeta_3^2 + \frac{2}{63}\pi^6 + 690\zeta_5 + \frac{269}{360}\pi^4 + \frac{52577}{36}\zeta_3 - \frac{14062811}{5184}\right)b^2 \\ + \left(1098\zeta_7 - \frac{8}{5}\pi^4\zeta_3 + 160\zeta_3^2 + \frac{20}{189}\pi^6 + \frac{95006}{15}\zeta_5 + \frac{2801}{1800}\pi^4 + \frac{198917}{30}\zeta_3 - \frac{39035933}{2430}\right)b^3 \\ + \left(240\zeta_3\zeta_5 + \frac{4}{225}\pi^8 + 21615\zeta_7 - \frac{16}{3}\pi^4\zeta_3 + \frac{397}{3}\zeta_3^2 + \frac{131}{567}\pi^6 \right. \\ \left. + \frac{838699}{18}\zeta_5 + \frac{14959}{10800}\pi^4 + \frac{34793081}{1080}\zeta_3 - \frac{51287121209}{466560}\right)b^4 + \dots \quad (4.17)$$

The b^3/β_0^2 term has canceled, so that the coefficient of C_F in the bracket in (2.7) is 0.

5. Conclusion

The terms with the highest powers of n_f at each order of perturbation theory ($C_F(T_F n_f)^{L-1} \alpha_s^L$ in Γ , γ_h ; $C_F(T_F n_f)^L \alpha_s^{L+1}$ in $V(\vec{q})$) are known, and given by explicit formulas (3.8), (3.10), (3.12). The terms with the next to highest power of n_f can have abelian (C_F^2) or non-abelian ($C_F C_A$) color

structure. The abelian terms $(C_F^2(T_F n_f)^{L-2} \alpha_s^L$ ($L \geq 3$) in Γ , γ_h ; $C_F^2(T_F n_f)^{L-1} \alpha_s^{L+1}$ ($L \geq 2$) in $V(\vec{q})$) are also known to all orders in α_s , but only as algorithms which allow one to obtain (in principle) any number of terms, see (4.13), (4.14), (4.16). The simple method used here is not applicable to non-abelian terms.

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